Coalescence in Galton-Watson Trees

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Outline

1. The Problem of Coalescence in Trees

Let $\mathcal{T}$ be a rooted tree. Let $\{v_{n1}, v_{n2}, \cdots, v_{nn}\}$ be the set of vertices at the nth level.

Pick two of the v_{ni}'s by SRSWOR (simple random sampling without replacement) (assuming $Z_n \geq 2$) and trace their lines of descent back in time till they meet for the first time. Call that generation X_n.

X_n is call the *coalescence time*.

1. The Problem of Coalescence in Trees

Problems:

a) Find the distribution of X_n.
b) Study its limit as $n \to \infty$.

X_n is also called the generation number of the LCA (Last common ancestor) or MRCA (Most recent common ancestor) etc.

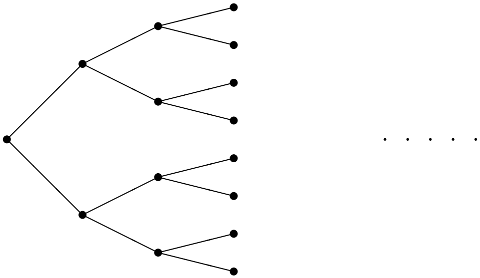
c) Do the same with choosing k vertices out of Z_n.
d) Do the same with choosing all Z_n vertices out of Z_n.

Clearly, the answers depend on how $\mathcal{T}$ is generated.

Outline

2. Binary Tree Case

Consider a binary tree $\mathcal{T}$ starting with one vertex. The tree looks like



At level n, there are 2^n vertices, $n = 0, 1, 2, \cdots$.

2. Binary Tree Case

Pick two vertices at level n by SRSWOR. Trace their lines back till they meet. Call that generation X_n. Then, for $k = 1, 2, \cdots, n$,

$$P(X_n < k) = \frac{\binom{2^k}{2} 2^{n-k} 2^{n-k}}{\binom{2^n}{2}} = \frac{2^k(2^k - 1)2^{n-k}2^{n-k}}{2^n(2^n - 1)} = \frac{1 - 2^{-k}}{1 - 2^{-n}}$$

So, $\lim_{n\to\infty} P(X_n < k) = 1 - 2^{-k}$, $k = 1, 2, \cdots$.

Thus, $X_n \xrightarrow{d} Geo(\frac{1}{2})$.

Similar result is true for any regular $b-$nary tree, $b \geq 2$.

This suggests that the same must be true for a growing Galton-Watson tree.

Outline

3.1 Definition and the problem

Let $\{p_j\}_{j\geq 0}$ be a probability distribution on $\mathbb{N}^+ \equiv \{0, 1, 2, \cdots\}$, $\{\xi_{n,i} : i \geq 1, n \geq 0\}$ be i.i.d $\sim \{p_j\}_{j\geq 0}$, Z_0 be a positive integer (r.v.),

$$Z_1 = \sum_{i=1}^{Z_0} \xi_{0,i}$$

and

$$Z_{n+1} = \begin{cases} \sum_{i=1}^{Z_n} \xi_{n,i} & , n \geq 0 \quad \text{if } Z_n > 0 \\ 0 & \text{if } Z_n = 0 \end{cases}$$

3.1 Definition and the problem

Then $\{Z_n\}_{n\geq 0}$ is called a Galton-Watson branching process with initial population Z_0 and offspring distribution $\{p_j\}_{j\geq 0}$, and $\xi_{n,i}$ is the number of offspring of the ith individual of the nth generation.
Now, every individual in the nth generation, $n \geq 1$, can be identified by a finite string

$$u_n \equiv (i_0, i_1, i_2, \cdots, i_n)$$

meaning that this individual is the i_nth offspring of the $u_{n-1} \equiv (i_0, i_1, \cdots, i_{n-1})$ and $u_0 = i_0$ is the number associated with the i_0th member of the 0th generation.

3.1 Definition and the problem

Let $A_{n,2} \equiv \{Z_n \geq 2\}$ and $B_n \equiv \{Z_n > 1\}$ be events defined on the space of trees.

Consider the following questions:

3.1 a) Conditioned on $A_{n,2}$, pick two individuals in the nth generation by SRSWOR and trace their lines back till they meet. Call that generation $X_{n,2}$.

What is the distribution of $X_{n,2}$?

What happens to it as $n \to \infty$?

3.1 Definition and the problem

3.1 b) Do the same thing with k choices ($2 \leq k < \infty$) by SRSWOR from the nth generation. Call the coalescence time $X_{n,k}$. Ask the same questions.

3.1 c) Do the same thing for the whole population. Call the coalescence time Y_n. Ask the same questions, i.e.,

What is the distribution of Y_n and what happens to it as $n \to \infty$?

3.2 Some basic results for Galton-Watson trees

3.2 i) (Supercritical case) Let $p_0 = 0$, $1 < m = \sum_{j=1}^{\infty} j p_j < \infty$.

Then

a) $P(Z_n \to \infty | Z_0 > 0) = 1$.

b) (Harris, 1960)

$$\left\{ W_n \equiv \frac{Z_n}{m^n} : n \geq 0 \right\}$$

is a nonnegative martingale and hence

$$\lim_{n \to \infty} W_n \equiv W \text{ exists w.p.1.}$$

3.2 Some basic results for Galton-Watson trees

3.2 i) (Supercritical case) Let $p_0 = 0$, $1 < m = \sum_{j=1}^{\infty} j p_j < \infty$.

Then

c) (Kesten and Stigum, 1966)

$$\sum_{j=1}^{\infty} (j \log j) p_j < \infty \quad \text{iff} \quad E(W | Z_0 = 1) = 1$$

and then W has an absolutely continuous distribution on $(0, \infty)$ with a positive density.

d) (Seneta and Heyde, 1970)

$$\exists C_n \ni \frac{C_{n+1}}{C_n} \to m \quad \text{and} \quad \frac{Z_n}{C_n} \to W \text{ w.p.1}$$

and $P(0 < W < \infty) = 1$.

3.2 Some basic results for Galton-Watson trees

3.2 i) (Supercritical case) Let $p_0 = 0$, $1 < m = \sum_{j=1}^{\infty} j p_j < \infty$.

Then

e) (Athreya and Schuh, 2003)

$$E(W : W \leq x) \equiv L(x)$$

is slowly varying at ∞.

3.2 Some basic results for Galton-Watson trees

3.2 ii) (Critical case) Let $m \equiv \sum_{j=1}^{\infty} j p_j = 1$, $p_j \neq 1$ for any $j \geq 1$ and $\sigma^2 \equiv \sum_{j=1}^{\infty} j^2 p_j - 1 < \infty$. Then

a) $P(Z_n \to 0 | Z_0 > 0) = 1$.

b) (Kolmogrov, 1938)

$$nP(Z_n > 0) \to \frac{\sigma^2}{2} \quad \text{as } n \to \infty.$$

c) (Yaglom, 1947)

$$P\left(\frac{Z_n}{n} > x \middle| Z_n > 0\right) \to e^{-\frac{2}{\sigma^2}x} \quad , 0 < x < \infty.$$

3.2 Some basic results for Galton-Watson trees

3.2 ii)

d) (Athreya, 2010) For $1 \leq k \leq n$, let

$$V_{n,k} \equiv \left\{ \frac{Z^{(k)}_{n-k,i}}{n-k} I_{(Z^{(k)}_{n-k,i}>0)} : 1 \leq i \leq Z_k \right\}$$

on the event $\{Z_k > 0\}$, where $\{Z^{(k)}_{j,i} : j \geq 0\}$ is the G-W process initiated by the ith individual in the kth generation. Let $k \to \infty$, $n \to \infty$ such that $\frac{k}{n} \to u$, $o < u < 1$.
Then the sequence of point processes $\{V_{n,k}\}_{n\geq 1}$ conditioned on $\{Z_n \geq 1\}$ converges weakly to the point process

$$V \equiv \{\eta_j : j = 1, 2, \cdots, N_u\}$$

where $\{\eta_j\}_{j\geq 1}$ are i.i.d. $\mathsf{exp}(1)$, N_u is $Geom(u)$, i.e., $P(N_u = k) = (1-u)u^{k-1}$, $k \geq 1$ and $\{\eta_j\}_{j\geq 1}$ and N_u are independent.

3.2 Some basic results for Galton-Watson trees

3.2 iii) (Subcritical case) (Yaglom, 1947) Let $0 < m \equiv \sum_{j=1}^{\infty} j p_j < 1$.

Then

a) For $j \geq 1$, $\lim_{n\to\infty} P(Z_n = j | Z_n > 0) \equiv b_j$ exists, $\sum_{j=0}^{\infty} b_j = 1$

and $B(s) \equiv \sum_{j=0}^{\infty} b_j s^j$, $0 \leq s \leq 1$ is the unique solution of the functional equation

$$B(f(s)) = mB(s) + (1 - s) \quad , 0 \leq s \leq 1$$

where $f(s) \equiv \sum_{j=0}^{\infty} p_j s^j$, in the class of all probability generating functions vanishing at 0.

3.2 Some basic results for Galton-Watson trees

3.2 iii) (Subcritical case) (Yaglom, 1947) Let $0 < m \equiv \sum_{j=1}^{\infty} j p_j < 1$.

Then

b) $$\sum_{j=1}^{\infty} j b_j < \infty \quad \text{iff} \quad \sum_{j=1}^{\infty} (j \log j) p_j < \infty.$$

c) $$\lim_{n\to\infty} \frac{P(Z_n > 0 | Z_0 = 1)}{m^n} = \frac{1}{\sum_{j=1}^{\infty} j b_j}.$$

3.2 Some basic results for Galton-Watson trees

3.2 iii) (Subcritical case) Let $0 < m \equiv \sum_{j=1}^{\infty} j p_j < 1$. Let Z_0 be a random variable. Then

d) If $EZ_0 < \infty$, then

$$\lim_{n\to\infty} P(Z_n = j | Z_n > 0) = b_j \quad , \forall j \geq 1$$

and if, in addition, $\sum_{j=1}^{\infty} (j \log j) p_j < \infty$ then

$$\sum_{j=1}^{\infty} j b_j < \infty \quad \text{and} \quad \lim_{n\to\infty} \frac{P(Z_n > 0)}{m^n} = \frac{EZ_0}{\sum_{j=1}^{\infty} j b_j}.$$

Outline

4.1 Supercritical case

Theorem 4.1:

Theorem

(Supercritical case) Let $p_0 = 0,\ 1 < m \equiv \sum_{j=1}^{\infty} j p_j < \infty$. *Then, for almost all trees* $\mathcal{T}$,

i) *for* $\forall 1 \leq k < \infty$,

$$\lim_{n\to\infty} P(X_{n,2} < k|\mathcal{T}) \equiv \pi_{k,2}(\mathcal{T}) \quad \textit{exists}$$

and $\pi_{k,2}(\mathcal{T}) \uparrow 1$ *as* $k \uparrow \infty$.

4.1 Supercritical case

Theorem 4.1:

Theorem

ii) *for* $\forall j \geq 2$, $\forall 1 \leq k < \infty$,

$$\lim_{n\to\infty} P(X_{n,j} < k|\mathcal{T}) \equiv \pi_{k,j}(\mathcal{T}) \quad \textit{exists}$$

and $\pi_{k,j}(\mathcal{T}) \uparrow 1$ *as* $k \uparrow \infty$.

iii) *Let* $p_1 > 0$. *Then, for almost all trees* $\mathcal{T}$,

$$Y_n \to N(\mathcal{T})$$

where $N(\mathcal{T}) = \max\{j \geq 1 : Z_j = 1\}$. *Also,*

$$\lim_{n\to\infty} P(Y_n = k) = (1 - p_1)p_1^k \quad , k \geq 0.$$

4.2 Critical case

Theorem 4.2:

Theorem

(Critical case) Let $m = 1$, $p_1 < 1$ *and* $\sigma^2 = \sum_{j=1}^{\infty} j^2 p_j - 1 < \infty$,

Then, for $0 < u < 1$,

i) $\lim_{n\to\infty} P\left(\frac{X_{n,2}}{n} \middle| Z_n \geq 2\right) \equiv H_2(u)$ *exists and for* $0 < u < 1$,

$$H_2(u) \equiv 1 - E\varphi(N_u)$$

where N_u *is a geometric random variable with distribution*

$$P(N_u = k) = (1-u)u^{k-1} \quad , k \geq 1$$

4.2 Critical Case

Theorem 4.2:

Theorem

i) *(continued) and for* $j \geq 1$,

$$\varphi(j) \equiv E\left(\frac{\sum_{i=1}^{j} \eta_i^2}{\left(\sum_{i=1}^{j} \eta_i\right)^2}\right)$$

where $\{\eta_i\}_{i\geq 1}$ *are i.i.d. exponential r.v. with* $E\eta_1 = 1$. *Further,* $H_2(\cdot)$ *is absolutely continuous on* $[0, 1]$, $H_2(0+) = 0$, *and* $H_2(1-) = 1$.

4.2 Critical Case

Theorem 4.2:

Theorem

ii) *for* $0 < u < 1$, $1 < k < \infty$,

$$\lim_{n\to\infty} P\left(\frac{X_{n,k}}{n} < u \middle| Z_n \geq k\right) \equiv H_k(u) \quad \textit{exists}$$

and $H_k(\cdot)$ *is an a.c. distribution function with* $H_k(0+) = 0$ *and* $H_k(1-) = 1$.

iii) *for* $0 < u < 1$, $\lim_{n\to\infty} P\left(\frac{Y_n}{n} < u \middle| Z_n \geq 1\right) = u$.

Remark: iii) above is also proved in Zubkov (1974) (TPA).

4.3 Subcritical case

Theorem 4.3:

Theorem

(Subcritical case) Let $0 < m \equiv \sum_{j=1}^{\infty} jp_j < 1$. *Then*

i) *For* $k \geq 1$, $\lim_{n\to\infty} P(n - X_n > k | Z_n \geq 2) = \dfrac{E\phi_k(Y)}{E\psi_k(Y)} \equiv \pi_k$, *say, where*

$$\phi_k(j) = E\left(\frac{\sum_{i_1 \neq i_2 = 1}^{j} Z_{k,i_1} Z_{k,i_2}}{\left(\sum_{i=1}^{j} Z_{k,i}\right)\left(\sum_{i=1}^{j} Z_{k,i} - 1\right)} I\left(\sum_{i=1}^{j} Z_{k,i} \geq 1\right) \right)$$

4.3 Subcritical case

Theorem 4.3:

Theorem

i) *(continued) and*

$$\psi_k(j) = P\left(\sum_{i=1}^{j} Z_{k,i} \geq 2\right)$$

where $\{Z_{r,i} : r \geq 0\}$, $i = 1, 2, \cdots$ *are i.i.d. copies of a Galton-Watson branching process* $\{Z_r : r \geq 0\}$ *with* $Z_0 = 1$ *and the given offspring distribution* $\{p_j\}_{j \geq 0}$ *and* Y *is a random variable with distribution* $\{b_j\}_{j \geq 1}$ *where*

$$b_j \equiv \lim_{n \to \infty} P(Z_n = j | Z_n > 0, Z_0 = 1) \quad \textit{which exists.}$$

4.3 Subcritical case

Theorem 4.3:

Theorem

i) *(continued) Further, if* $\sum_{j=1}^{\infty} j \log j p_j < \infty$*, then* $\lim_{k\uparrow\infty} \pi_k = 0$ *and hence* $n - X_n$ *conditioned on* $Z_n \geq 2$ *converges to a proper distribution on* $\{1, 2, \cdots\}$.

ii) *For* $k \geq 1$, $\lim_{n\to\infty} P(n - Y_n > k | Z_n \geq 1) \equiv \tilde{\pi}_k$ *exists and equals*

$$E\left(\frac{1 - q_k^Y}{m^k}\right) - E\left(\frac{Y q^{k-1}(1 - q_k)}{m^k}\right)$$

4.3 Subcritical case

Theorem 4.3:

Theorem

ii) *(continued) where Y is a random variable with distribution*

$$P(Y = j) = b_j = \lim_{n\to\infty} P(Z_n = j | Z_n > 0, Z_0 = 1)$$

and $q_k = P(Z_k = 0 | Z_0 = 1)$.

Further, if $\sum_{j=1}^{\infty} j \log j p_j < \infty$, *then* $\lim_{k\to\infty} \tilde{\pi}_k = 0$. *That is,* $n - Y_n$ *conditioned on* $\{Z_n > 0\}$ *converges in distribution as* $n \to \infty$ *to a proper distribution on* $\{1, 2, \cdots\}$.

See also A. Lambert AAP (2003) 35, 1071-189.

4.4 Explosive case

Theorem 4.4:

Theorem

(Explosive case) Let $p_0 = 0$, $m = \sum_{j=1}^{\infty} j p_j = \infty$, *and for some* $0 < \alpha < 1$, *and a function* $L : (1, \infty) \to (0, \infty)$ *slowly varying at* ∞, *i.e.,* $\forall 0 < c < \infty$,

$$\frac{L(cx)}{L(x)} \to 1 \quad \text{as } x \to \infty.$$

Let

$$\frac{\sum_{j>x} p_j}{x^{\alpha} L(x)} \to 1 \quad \text{as } x \to \infty.$$

4.4 Explosive case

Theorem 4.4:

Theorem

(continued) Then

i) *(Davies, 1979)* $\alpha^n \log Z_n \to \eta$ *w.p.1 and* $P(0 < \eta < \infty) = 1$ *and* η *has a continuous distribution.*

ii) *(Grey, 1980) Let* $\{Z_n^{(1)}\}_{n \geq 1}$ *and* $\{Z_n^{(2)}\}_{n \geq 1}$ *be two i.i.d. copies of a GWBP with* $\{p_j\}_{j \geq 1}$ *satisfying the above hypotheses. Then, w.p.1*

$$\frac{Z_n^{(1)}}{Z_n^{(2)}} \to \begin{cases} 0 & \text{with prob. } \frac{1}{2} \\ \infty & \text{with prob. } \frac{1}{2} \end{cases}$$

4.4 Explosive case

Theorem 4.4:

Theorem

(continued)

iii) *For almost all trees $\mathcal{T}$ and $k = 1, 2, \cdots$, as $n \to \infty$,*

$$P(X_{n,2} < k | \mathcal{T}) \to 0$$

and

$$P(n - X_{n,2} < k) \to \pi_2(k) \quad \textit{exists}$$

and $\pi_2(k) \uparrow 1$ *as* $k \uparrow \infty$.

4.4 Explosive case

Theorem 4.4:

Theorem

(continued)

iv) *For any* $1 < j < \infty$ *and* $k = 1, 2, \cdots$

$$P(X_{n,j} < k|\mathcal{T}) \to 0 \quad \text{as } n \to \infty$$

and $P(n - X_{n,j} < k) \to \pi_j(k)$ *exists and* $\pi_j(k) \uparrow 1$ *as* $k \uparrow \infty$.

v) $Y_n \xrightarrow{d} N(\mathcal{T}) \equiv \max\{j : Z_j = 1\} < \infty$ *and*

$$P(Y_n = k) \to (1 - p_1)p_1^{k-1} \quad , k \geq 1.$$

Proposition 4.1

The proof of Theorem 4.4 ($m = \infty$ explosive case) needs the following results.

Proposition

Let $\{Z_n\}_{n \geq 0}$ be a GWBP with offspring distribution $\{p_j\}_{j \geq 0} \in D(\alpha)$, (domain of attraction of a stable law of order α), $0 < \alpha < 1$, and $Z_0 = 1$. Then,

$$Z_k \in D(\alpha^k) \quad \forall 1 \leq k < \infty.$$

Proposition 4.2

Proposition

(Lepage, Woodroffe, Zinn, Ann. Prob., 1980)
Let $\{X_i\}_{i\geq 1}$ *be i.i.d. random variables s.t.* $P(0 < X_1 < \infty) = 1$ *and* $X_1 \in D(\alpha)$, $0 < \alpha < 1$. *Then*

a)

$$\frac{\sum_{i=1}^{n} X_i^2}{\left(\sum_{i=1}^{n} X_i\right)^2} \xrightarrow{d} Y_\alpha$$

where Y_α *is a continuous r.v. with* $P(0 < Y_\alpha < 1) = 1$.

Proposition 4.2

Proposition

(continued)

b) $EY_\alpha \uparrow 1$ *as* $\alpha \downarrow 0$.

c) *For any* $j = 2, 3, \cdots$,

$$\frac{\sum_{i=1}^{n} X_i^j}{\left(\sum_{i=1}^{n} X_i\right)^j} \xrightarrow{d} Y_{\alpha,j}$$

and $EY_{\alpha,j} \uparrow 1$ *as* $\alpha \downarrow 0$.

Basic Calculation

$$P(X_n \geq k|\mathcal{T}) = \frac{\sum_{i=1}^{Z_k} \binom{Z^{(k)}_{n-k,i}}{2}}{\binom{Z_n}{2}}$$

$$= \frac{\sum_{i=1}^{Z_k} Z^{(k)}_{n-k,i}\left(Z^{(k)}_{n-k,i} - 1\right)}{\left(\sum_{i=1}^{Z_k} Z^{(k)}_{n-k,i}\right)\left(\sum_{i=1}^{Z_k} Z^{(k)}_{n-k,i} - 1\right)} \qquad (*)$$

Basic Calculation

a) $1 < m < \infty$

Fix k, by Seneta-Heyde, $\exists C_n \ni$

$$\frac{Z^{(k)}_{n-k,i}}{m^{n-k}} \to W_{k,i} \quad \text{w.p.1}$$

and $P(0 < W_{k,i} < \infty) = 1$. So,

$$(*) \to \frac{\sum_{i=1}^{Z_k} W_{k,i}^2}{\left(\sum_{i=1}^{Z_k} W_{k,i}\right)^2}$$

and this converges to 0 as $k \to \infty$ by O'Brien's theorem (1980):

Basic Calculation

a) (continued)
Let $\{X_i\}_{i \geq 1}$ be i.i.d. positive random variables s.t. $E(X_1 : X_1 \leq x)$ is slowly varying at ∞. Then

$$\frac{\max_{1 \leq i \leq n} X_i}{\sum_{i=1}^{n} X_i} \xrightarrow{p} 0.$$

Basic Calculation

b) $m = \infty$, $\{p_j\} \in D(\alpha)$, $0 < \alpha < 1$.

$$\begin{aligned} P(n - X_n \leq k) &= P(X_n \geq n - k) \\ &= E\left(\frac{\sum_{i=1}^{Z_{n-k}} Z_{k,i}^{(n-k)} \left(Z_{k,i}^{(n-k)} - 1\right)}{Z_n(Z_n - 1)} \right) \\ &\to \pi(k) \equiv E\left(Y_{\alpha,k}\right) \end{aligned}$$

by Lepage, Woodroff and Zinn, and $\pi(k) \uparrow 1$ as $k \uparrow \infty$ and $E\left(Y_\alpha\right) \uparrow 1$ as $\alpha \downarrow 0$.

c) Similar argument for $m = 1$ and $0 < m < 1$. (need point process result for $m = 1$ and the Yaglom theorem for $0 < m < 1$).

Summary

$1 < m < \infty$: $X_{n,2} \xrightarrow{d}$ a proper distribution on $\{0, 1, 2, \cdots\}$

$m = \infty$, $\{p_j\}_{j \geq 0} \in D(\alpha)$, $0 < \alpha < 1$: $n - X_{n,2} \xrightarrow{d}$ a proper distribution on $\{0, 1, 2, \cdots\}$

$m = 1$, $\sigma^2 < \infty$: $\left.\frac{X_{n,2}}{n}\right| Z_n \geq 2 \xrightarrow{d}$ a.c. distribution on $[0, 1]$

$\left.\frac{Y_n}{n}\right| Z_n \geq 1 \xrightarrow{d}$ uniform distribution on $[0, 1]$

$0 < m < 1$: $\left.(n - X_{n,2})\right| Z_n \geq 2 \xrightarrow{d}$ a proper distribution on $\{1, 2, \cdots\}$

Summary

i.e.

$1 < m < \infty$: coalescence is **near the beginning of the tree**.

$m = \infty$, $\{p_j\}_{j \geq 0} \in D(\alpha)$, $0 < \alpha < 1$: coalescence is **near the present**.

$m = 1$, $\sigma^2 < \infty$: $X_{n,2}$ is **of order** n.

$0 < m < 1$: $X_{n,2}$ is **near the present**.

Outline

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   - Basic results
4. Coalescence results for Galton-Watson trees
   - Supercritical ($1 < m < \infty$)
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   - Explosive ($m = \infty$, $\{p_j\} \in D(\alpha)$, $0 < \alpha < 1$)
5. Branching random walks
   - $1 < m < \infty$
   - $m = \infty$, $\{p_j\} \in D(\alpha)$, $0 < \alpha < 1$
6. Scaling limits of Bellman-Harris Processes with age dependent Markov motion: Supercritical and critical cases

5. Branching Random Walks

Let $\mathcal{T}$ be a G-W tree with $Z_0 = 1$ and offspring distribution $\{p_j\}_{j \geq 0}$.

Impose on this tree $\mathcal{T}$ the following movement structure:

If an individual is at x in $\mathbb{R}$ and has k children then these k children move to $x + X_{k,j}$, $j = 1, 2, \cdots, k$, where $X_k \equiv (X_{k,1}, X_{k,2}, \cdots, X_{k,k})$ has a joint distribution $\pi_k(\cdot)$ on $\mathbb{R}^k$.

Also, the random vector X_k is stochastically independent of the history up to that generation as well as the movement of the other individuals of that generation.

5. Branching Random Walks

Let Z_n be the number of individuals in the nth generation and $\zeta_n \equiv \{x_{n,i} : 1 \leq i \leq Z_n\}$ be the positions of the Z_n individuals of the nth generation.

A problem of interest is what happens to the point process ζ_n as $n \to \infty$.

Theorem 5.1

Theorem

Let $p_0 = 0$, $1 < m \equiv \sum_{j=1}^{\infty} j p_j < \infty$ *and* π_k *be such that* $\{X_{k,i} : i = 1, 2, \cdots, k\}_{k \geq 1}$ *are identically distributed.*

a) *Let* $EX_{k,1} = 0$ *and* $EX_{k,1}^2 = \sigma^2 < \infty$. *Then,* $\forall y \in \mathbb{R}$,

$$\frac{Z_n(\sqrt{n}\sigma y)}{Z_n} \to \Phi(y) \quad (\textit{the standard } N(0,1) \textit{ cdf})$$

in mean square.

Theorem 5.1

Theorem

(continued)

b) *If* $X_{k,1} \in D(\alpha)$, $0 < \alpha \leq 2$, *then* $\exists\ a_n,\ b_n \ni$

$$\frac{Z_{a_n+b_n y}}{Z_n} \to G_\alpha(y) \quad \text{in mean square,}$$

where $G_\alpha(\cdot)$ *is a standard stable law cdf (of order* α*).*

c) *In a), if* Y_n *is the position of a randomly chosen individual from the nth generation, then,* $\forall y \in \mathbb{R}$,

$$P(Y_n \leq \sqrt{n}\sigma y) \to \Phi(y)$$

and similarly for b).

Theorem 5.1

The proof depends on the fact when $p_0 = 0$ and $1 < m \equiv \sum_{j=1}^{\infty} jp_j < \infty$, the coalescence time $X_{n,2}$ is way back in time and so the positions of two randomly chosen individuals in the nth generation are essentially independent and has the marginal distribution of a random walk at step n.

Theorem 5.2

Theorem

(Athreya-Hong, 2011)
Let $m = \infty$, $\{p_j\}_{j \geq 0} \in D(\alpha)$, $0 < \alpha < 1$. *Let* $\{X_{k,i} : 1 \leq i \leq k\}_{k \geq 1}$ *be identically distributed. Let* $EX_{k,1} = 0$ *and* $EX_{k,1}^2 = \sigma^2 < \infty$. *Then*

$$\frac{Z_n(\sqrt{n}\sigma y)}{Z_n} \xrightarrow{d} \delta_y$$

where δ_y *is Bernoulli*$(\Phi(y))$, *i.e.*

$$\delta_y = \begin{cases} 1, & \text{with prob. } \Phi(y) \\ 0, & \text{with prob. } 1 - \Phi(y) \end{cases}$$

Theorem 5.2

The proof is based on the fact that

$$E\left(\frac{Z_n(\sqrt{n}\sigma y)}{Z_n}\right)^k \to \Phi(y) \quad \text{for } k = 1, 2.$$

This, in turn, is due to the fact that $X_{n,2}$, the coalescence time for any two individuals chosen at random from the nth generation is such that $n - X_{n,k}$ converges to a proper distribution (Theorem 4.4) and hence their positions differ only by an amount that converges in distribution.

This can be strengthened to joint convergence of

$$\frac{Z_n(\sqrt{n}\sigma y)}{Z_n}, \quad i = 1, 2, \cdots, k$$

Theorem 5.3

Theorem

(Athreya-Hong, 2011)

Under the hypothesis of Theorem 5.2,

a) *for any* $-\infty < y_1 < y_2 < \infty$,

$$\left(\frac{Z_n(\sqrt{n}\sigma y_1)}{Z_n}, \frac{Z_n(\sqrt{n}\sigma y_2)}{Z_n}\right) \xrightarrow{d} (\delta_1(\Phi(y_1)), \delta_2(\Phi(y_2)))$$

which takes values $(0,0)$, $(0,1)$ *and* $(1,1)$ *with probabilities* $1 - \Phi(y_2)$, $\Phi(y_2) - \Phi(y_1)$ *and* $\Phi(y_1)$, *respectively.*

Theorem 5.3

Theorem

(continued)

b) *for any* $-\infty < y_1 < y_2 < \cdots < y_k < \infty$,

$$\left(\frac{Z_n(\sqrt{n}\sigma y_i)}{Z_n} : 1 \leq i \leq k\right) \xrightarrow{d} (\delta_1, \cdots, \delta_k))$$

where each δ_i *is* 0 *or* 1 *and further* $\delta_i = 1 \Rightarrow \delta_j = 1$ *for* $j \geq i$ *and*

$$\begin{aligned} & P(\delta_1 = 0, \delta_2 = 0, \cdots, \delta_{j-1} = 0, \delta_j = 1, \cdots, \delta_k = 1) \\ = \;& P(\delta_{j-1} = 0, \delta_j = 1) = \Phi(y_j) - \Phi(y_{j-1}). \end{aligned}$$

Theorem 5.3

This suggests that

$$\left\{ Z_n(y) = \frac{Z_n(\sqrt{n}\sigma y)}{Z_n}, -\infty < y < \infty \right\}$$

converges in the Skorohod Space $D(-\infty, \infty)$ weakly to

$$\{X(y) \equiv I_{N \le y}, -\infty < y < \infty\}$$

where N is a $N(0,1)$ r.v.

This needs to be proved. Only tightness needs to be established.

Theorem 5.4

Theorem

If Y_n is the position of a randomly chosen individual in the nth generation, then in all cases (as long as $p_0 = 0$), given the tree (random walk) $\mathcal{T}$, $\forall y \in \mathbb{R}$,

$$P(Y_n \le \sqrt{n}\sigma y | \mathcal{T}) \xrightarrow{d} \delta_y \sim Ber(\Phi(y))$$

This is so since

$$P(Y_n \le \sqrt{n}\sigma y | \mathcal{T}) = \frac{Z_n(\sqrt{n}\sigma y)}{Z_n}$$

and this in turn implies, $\forall y \in \mathbb{R}$,

$$P(Y_n \le \sqrt{n}\sigma y) \to \Phi(y).$$

Remark 1

Remark

Theorem 5.1 holds under the following weaker assumption about π_k, the distribution of $(X_{k,1}, X_{x,2}, \cdots, X_{k,k})$, that does not require $\{X_{k,1}\}_{k\geq 1}$ to be identically distributed. It suffices to assume:

i) $\forall k \geq 1$, $(X_{k,1}, X_{x,2}, \cdots, X_{k,k})$ has a distribution that is invariant under permutation.

ii) If $\{p_k\}_{k\geq 1}$ is the offspring distribution with

$$\sum_{k=1}^{\infty} p_k E X_{k,1}^2 < \infty, \quad 1 < m = \sum_{k=1}^{\infty} k p_k < \infty, \quad p_0 = 0.$$

Theorem 5.1′

Now let $\mu = \sum_{k=1}^{\infty} p_k E X_{k,1} < \infty$, $\sigma^2 = \sum_{k=1}^{\infty} p_k E X_{k,1}^2 - \mu^2$.

Theorem

Let $\zeta_n \equiv \{x_{n,1}, x_{n,2}, \cdots, x_{n,Z_n}\}$ be as in Theorem 5.1. Under the above assumptions, the following holds: for $\forall y \in \mathbb{R}$,

$$\frac{Z_n(n\mu + y\sigma\sqrt{n})}{Z_n} \equiv \frac{1}{Z_n} \sum_{i=1}^{Z_n} I(x_{n,i} \le n\mu + y\sigma\sqrt{n}) \to \Phi(y) \quad \textit{in mean square.}$$

Application to energy cascades

Consider a particle that under goes fission.

Assume each particle spits into a random number of new particles with distribution $\{p_k\}_{k \geq 1}$.

Assume that the energy x of the parent is split to $\{xY_{k,1}, xY_{k,2}, \cdots, xY_{k,k}\}$ for each of the offspring particle if the parent splits into k offspring particles.

Application to energy cascades

Then the energy e_{n,I_n} of a particle I_n in the nth generation can be represented as

$$x_0 Y_{u_1} Y_{u_2} \cdots Y_{u_n}$$

where u_n, $u_{n-1}, \cdots$, u_1 are the addresses of the individual I_n and its ancestors and x_0 is the energy of the ancestor 1.

Assume Y_{u_i}'s are independent. Clearly, the distribution of Y_{u_i} depends on the number of offspring of individual u_{i-1} and

$$\left\{ \log e_{n,I_n}, I_n \in n\text{th generation} \right\}$$

is a branching random walk.

Theorem 5.2″

So, from Theorem 5.1′, one gets the following.

Theorem

Let $\{X_{k,i} \equiv \log Y_{k,i} : 1 \leq i \leq k\}_{k\geq 1}$ *and* $\{p_k\}_{k\geq 1}$ *satisfy the conditions of Theorem 5.1′. Then,* $\forall y \in \mathbb{R}$*, as* $n \to \infty$*,*

$$\frac{Z_n(n\mu + y\sigma\sqrt{n})}{Z_n} \equiv \frac{1}{Z_n}\sum_{i=1}^{Z_n} I(\log e_{n,i} \leq n\mu + y\sigma\sqrt{n})$$

$$\to \Phi(y) \quad \textit{in mean square.}$$

Open Cases

Open Cases: $m = 1$ and $0 < m < 1$.

Outline

1. The Problem of Coalescence in Trees
2. Binary Tree Case
3. Galton-Watson Tree Case
   - Definition
   - Basic results
4. Coalescence results for Galton-Watson trees
   - Supercritical ($1 < m < \infty$)
   - Critical ($m = 1$)
   - Subcritical ($0 < m < 1$)
   - Explosive ($m = \infty$, $\{p_j\} \in D(\alpha)$, $0 < \alpha < 1$)
5. Branching random walks
   - $1 < m < \infty$
   - $m = \infty$, $\{p_j\} \in D(\alpha)$, $0 < \alpha < 1$
6. Scaling limits of Bellman-Harris Processes with age dependent Markov motion: Supercritical and critical cases

Scaling Limits of B-H processes with age dependent Markov motion

Suppose we are given:

i) an offspring distribution $\{p_j\}_{j\geq 1}$ on $\mathbb{N}^+ \equiv \{0, 1, 2, \cdots\}$
ii) a lifetime distribution $G(\cdot)$ on $(0, \infty)$ and non-latice
iii) a real-valued Markov process $\eta(\cdot)$ on $[0, \infty)$ with $\eta(0) = 0$

First, generate a BH tree $\mathcal{T}$ with offspring distribution $\{p_j\}_{j\geq 0}$ and lifetime distribution $G(\cdot)$ and an initial population at $t = 0$ of size Z_0.

Now, suppose that the initial population is located at $x_{0,i}$, $i = 1, 2, \cdots, Z_0$ and with ages $a_{0,i}$, $i = 1, 2, \cdots, Z_0$.

Scaling Limits of B-H processes with age dependent Markov motion

Assume each individual moves during its lifetime of length L according to Markov process $\{x + \eta(t) : 0 \leq t \leq L\}$.

That is, if an individual is born at time τ and at location x and has lifetime L, then its movement

$$\{X(t) : \tau \leq t < \tau + L\}$$

is distributed as

$$\{x + \eta(t - \tau) : \tau \leq t < \tau + L\}$$

where $\{\eta(\cdot)\}$ is a real-valued Markov process on $[0, \infty)$ with $\eta(0) = 0$.

Scaling Limits of B-H processes with age dependent Markov motion

Assume that, for each individual, the lifetime L, the number of offspring ξ and the movement process $\eta(\cdot)$ are independent and the triplets (L, ξ, η) over all the individuals in the tree are i.i.d.

Let Z_t be the population size at time t and

$$C_t \equiv \left\{(a_{t,i}, x_{t,i}) : 1 \leq i \leq Z_t\right\}$$

be the age and position configuration of all the individuals alive at time t.

The object of study is the point process $\{C_t : t \geq 0\}$.

Theorem 6.1

Theorem

(Supercritical case) (Athreya-Athreya-Iyer, Bernoulli 2011)

Let $p_0 = 0,\ 1 < m \equiv \sum_{j=1}^{\infty} j p_j < \infty$. *Let* $E\eta(0) \equiv 0$,

$v(t) \equiv E\eta^2(t) < \infty,\ \sup_{0 \le s \le t} v(s) < \infty$ *and*

$$\psi_\alpha \equiv \int_{[0,\infty)} e^{-\alpha s} v(s) dG(s) < \infty$$

where $0 < \alpha < \infty$ *is the Malthusian parameter defined by*

$$m \int_{[0,\infty)} e^{-\alpha s} dG(s) = 1.$$

Theorem 6.1

Theorem

(continued) Let (a_t, X_t) be the age and position of a randomly chosen individual at time t. Then

a)

$$\left(a_t, \frac{X_t}{\sqrt{t}}\right) \xrightarrow{d} (U, V)$$

where U and V are independent and U has pdf proportional to $e^{-\alpha x}\left(1-G(x)\right)$ on $(0, \infty)$ and V is $N\left(0, \frac{\psi_\alpha}{\mu_\alpha}\right)$ where $\mu_\alpha = m\int_0^\infty xe^{-\alpha x dG(x)}$.

Theorem 6.1

Theorem

(continued)

b) *Let*

$$Y_y(A \times B) = \frac{1}{Z_t} \sum_{i=1}^{Z_t} I_{A\times B}\big(a_{t,i}, \frac{x_{t,i}}{\sqrt{t}}\big)$$

be the scaled empirical measure of $C_t \equiv \big\{(a_{t,i}, x_{t,i}) : 1 \le i \le Z_t\big\}$.
Then, $Y_t \xrightarrow{d} (U, V)$, *where* U *and* V *are as in a).*

The proof of this depends on the following results of independent interest.

Proposition 6.1

Proposition

Let M_t be the generation number of a randomly chosen individual from Z_t (those alive at time t). Let $\{L_{t,i} : 1 \leq t \leq M_t\}$ be the lifetimes of the ancestors of this individual. Then

a) *as $t \to \infty$,*

$$\frac{M_t}{t} \to \frac{1}{\mu_\alpha} \quad w.p.1.$$

Proposition 6.1

Proposition

(continued)

b) *for any* $h : [0,\infty) \to \mathbb{R}$ *Borel measurable and* $\int_{[0,\infty)} |h(x)| e^{-\alpha x} dG(x) < \infty$, $0 < \alpha < \infty$,

$$P\left(\left| \frac{1}{M_t} \sum_{i=1}^{M_t} h(L_{t,i} - c_\alpha(h) \right| > \epsilon \right) \to 0 \quad \text{as } t \to \infty.$$

where $c_\alpha(h) = m \int_{[0,\infty)} h(x) e^{-\alpha x} dG(x)$.

Proposition 6.2

Both these results depend on a size-biasing estimate of a large deviation result, namely,

Proposition

Let $\{N(t) : t \geq 0\}$ *be a renewal process generated by* G. *Let* $1 < m < \infty$ *and* α *be the Malthusian parameter, i.e.,* $m \int_{[0,\infty)} e^{-\alpha x} dG(x) = 1$. *Then, for* $\forall \epsilon > 0$,

$$e^{-\alpha t} E\left(m^{N(t)} I\left(\left|\frac{N(t)}{t} - \frac{1}{\mu_\alpha}\right| > \epsilon\right)\right) = 0$$

where $\mu_\alpha = m \int_0^\infty x e^{-\alpha x} dG(x)$.

Proposition 6.2

Note that since

$$\frac{N(t)}{t} \to \frac{1}{\mu} \quad \text{w.p.1}$$

where $\mu = \int_{[0,\infty)} x dG(x)$, the event

$$\left| \frac{N(t)}{t} - \frac{1}{\mu_\alpha} \right| > \epsilon$$

is an event of large deviation.

Proposition 6.3

Proposition

(Coalescence time for BH process) (Athreya-Hong, 2011) Choose two individuals from those alive at time t at random by SRSWOR and trace their lines back in time to find the time of death $\tau_{t,2}$ of their last common ancestor. Let $p_o = 0$, $1 < m = \sum_{j=1}^{\infty} jp_j < \infty$. Then, for $0 < s < \infty$,

$$\lim_{t \to \infty} P(\tau_{t,2} < s) = H(s) \quad \textit{exists}$$

and $H(\cdot)$ is an absolutely continuous d. f. on $(0, \infty)$ with $H(0) = 0$, $H(\infty) = 1$.

Proposition 6.3

Same is true for the coalescence of r individuals chosen at random from those alive at time t (for $1 < r < \infty$).

However, the coalescence time for the whole population goes back to the beginning.

Open problems: Extend the results of Theorem 5.2 (BRW with $m = \infty$, $\{p_j\}_{j \geq 0} \in D(\alpha)$, $0 < \alpha < 1$) to the present setting.

Theorem 6.2

Theorem

(Critical case) Let $m = 1$, $\sum_{j=1}^{\infty} j^2 p_j < \infty$, $E\eta(t) \equiv 0$, $v(t) = E\eta^2(t) < \infty$, $\sup_{0 \le s \le t} v(s) < \infty$, $\forall t$, *and*

$$\psi = \int_{[0,\infty)} v(s) dG(s) < \infty.$$

Let $A_t \equiv \{Z_t > 0\}$. *Then,*

Theorem 6.2

Theorem

(continued) conditioned on A_t, the random vector

$$\left(a_t, \frac{X_t}{t}\right)$$

for a randomly chosen individual converges as $t \to \infty$ in distribution to (U, V) where U and V are independent with U having a pdf $\frac{1}{\mu}(1 - G(\cdot))$ on $(0, \infty)$ and $V \sim N\left(0, \frac{\psi}{\mu}\right)$.

Theorem 6.3

Theorem

Assume the hypothesis of Theorem 6.2. Then, conditioned on $A_t \equiv \{Z_t > 0\}$, *the empirical measure*

$$Y_t(A \times B) \equiv \frac{1}{Z_t} \sum_{i=1}^{Z_t} I_{A \times B}\left(a_{t,i}, \frac{X_{t,i}}{\sqrt{t}}\right)$$

converges as $t \to \infty$ *in distribution to a* *random measure* ν *characterized by its moment sequence*

$$m_k(\varphi) \equiv E\left(\langle \nu, \varphi \rangle\right)^k$$

where $\varphi \in C_b^+(\mathbb{R}^+ \times \mathbb{R})$.

Theorem 6.3

The $m_k(\varphi)$ can be expressed in terms of the coalescence times of k randomly chosen individuals alive at time t.

The proof depends on the following results.

Proposition 6.4

Proposition

Let $m = 1$, $\sum_{j=1}^{\infty} j^2 p_j < \infty$, $G(\cdot)$ *non-latice. Then*

i) $\forall\ \epsilon > 0$

$$P\left(\left|\frac{M_t}{t} - \frac{1}{\mu}\right| > \epsilon \middle| Z_t > 0\right) \to 0 \quad as\ t \to \infty.$$

Proposition 6.4

Proposition

(continued)

ii) *the coalescence time* $\tau_{2,t}$ *of two randomly chosen individuals from time* t *(conditioned on* $Z_t > 0$*) satisfies*

$$\lim_{t\to\infty} P\left(\frac{\tau_{2,t}}{t} \leq x \middle| Z_t > 0\right) = H(x) \quad \textit{exists}$$

for all $0 \leq x \leq 1$.

iii) *A similar result for the convergence of coalescence of* k *individuals.*

Remark

Note that, in the supercritical case ($1 < m < \infty$, $p_0 = 0$) BH process, $\tau_{2,t}$ converged to a proper distribution as $t \to \infty$.

And, in the critical case, $\frac{\tau_{2,t}}{t}$ conditioned on $Z_t > 0$ converges in distribution. That is , $\tau_{2,t}$ is of order t.

Related work: Lambert, Legall

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Thank You